

Relative Locality in κ -Poincaré

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Abstract. We show that the κ -Poincaré Hopf algebra can be interpreted in the framework of curved momentum space leading to the relativity of locality [1].

We study the geometric properties of the momentum space described by κ -Poincaré, and derive the consequences for particles propagation and energy-momentum conservation laws in interaction vertices, obtaining for the first time a coherent and fully workable model of the deformed relativistic kinematics implied by κ -Poincaré.

We describe the action of boost transformations on multi-particles systems, showing that in order to keep covariant the composed momenta it is necessary to introduce a dependence of the rapidity parameter on the particles momenta themselves. Finally, we show that this particular form of the boost transformations keeps the validity of the relativity principle, demonstrating the invariance of the equations of motion under boost transformations.

1. Introduction

Recently a proposal for the description of physics in a “classical-non gravitational” quantum gravity regime has been put forward [1, 2], in which spacetime is flat but momentum space is curved. This proposal was the result of a deepening in the understanding of the fate of the locality principle in quantum gravity-motivated generalizations of Special Relativity [3, 4, 5, 6]. The possibility of modifying only the momentum space and not the spacetime arises since this regime is characterized by negligible \hbar and G (we set the speed of light $c = 1$), so that both quantum and gravitational effects are small, with their ratio kept fixed, so that we do have an energy scale ($M_p \sim \sqrt{\hbar/G}$), but we don’t have a length scale ($L_p \sim \sqrt{\hbar G} \rightarrow 0$) governing the modifications of standard physics. This is the framework in which a generalization of special relativity emerges, in which the fact that two events take place in the same spacetime point is no more an absolute concept, and can only be established by observers close to the events themselves (relativity of locality).

On the other hand, there is a fairly large agreement on the expectation that quantum gravity might require some kind of generalization of Riemannian geometry to describe spacetime at the Planck-scale [7, 8, 9, 10]. One way of doing this takes the standpoint of generalizing the structures that describe the symmetries of spacetime, extending the concept of Lie groups to Quantum Groups, also known as Hopf algebras. The Quantum Group that attracted more interest in relation to Planck-scale physics is a dimensionful deformation of the Poincaré group, known as κ -Poincaré, where κ is a constant with dimensions of an energy, setting the deformation scale. Deforming in a dimensionful way the Poincaré group requires the introduction of just an energy scale, and not a length scale, in the theory. A length scale may at most emerge through the further introduction of the Planck constant, \hbar , and so it should be available only when κ -Poincaré is used to describe the symmetries of a quantum system. κ -Poincaré is also believed to only be able to describe the symmetries of physical systems in which spacetime curvature is absent [11, 12, 13], and so G can be neglected. Therefore it appears that κ -Poincaré may be relevant in a situation where physics depends on an energy scale, but both gravitational and quantum effects are small. This is exactly the regime where the principle of Relative Locality is thought to hold.

In this paper we show that actually κ -Poincaré can be interpreted in the framework of curved momentum space leading to the relativity of locality. In the framework developed in [1] different aspects of momentum space geometry, such as the metric, curvature, torsion and non-metricity, reflect into different kinematical and dynamical properties of the motion and interactions of particles. Therefore the Relative Locality construction allows us to deduce the physical implications of κ -Poincaré in a simple model whose physical interpretation is clear, which is what has been missed the most since the discovery of this Hopf algebra.

In the next Section we briefly review (based on the work reported in [1]) the physical implications of the geometrical properties of momentum space emerging in the Relative

Locality framework.

In Section 3 we show how the translation sector of the κ -Poincaré Hopf algebra can be used to represent the coordinates over a momentum space, establishing a general correspondence between commutative Hopf algebras and the geometric structures introduced in Ref. [1], in a way that can be applied also to other Hopf algebras.

In the following Section 4 we are able to show that κ -Poincaré describes a momentum space with de Sitter metric and with torsion and nonmetricity. We can then follow the prescriptions given in Section 2 to make the connection between the geometrical properties of this momentum space and the physics that it describes. A byproduct of our analysis are the identification of a dispersion relation that is natural from the perspective of the geometry of momentum space (it is the geodesic distance from the origin), and such that the mass corresponds to the particle's rest energy.

Within this interpretation of κ -Poincaré it is possible to show (and we do this in Section 5) that Lorentz transformations act nonlinearly on momenta, and, even more curiously, that they have to act differently on different momenta, when they belong to an interaction vertex, in order to keep covariant the composed momenta. In particular, we find that the rapidity parameter for each momentum is subject to a “back-reaction” by the momenta of the other interacting particles.

In section 6 we show that in this physical framework the relativity principle still holds, in the sense that the equations of motion are invariant under boost transformations. This result is particularly relevant because in the past the issue of whether κ -Poincaré implies a breakdown of the relativity principle was subject to debate [14]. Of course, the interest of κ -Poincaré as an algebra of physical symmetries would be seriously reduced, if it turned out that it implied the breakdown of those symmetries. Our result provides the first explicit example of how the equivalence between inertial observers is realized in the context of Relative Locality, and it turns out to be realized in a particularly nontrivial way.

In Section 7 we identify a structure, related to the tangent space at the origin of momentum space, which reproduces the commutation relations of κ -Minkowski, a noncommutative spacetime whose symmetries are thought to be described by κ -Poincaré [15]. This result suggests that such a noncommutative space could emerge upon quantization of certain (space-time) degrees of freedom of our model.

2. Preliminaries on the Relative Locality principle

The relativity of locality is achieved in [1] by assuming the phase space as the fundamental arena where physics takes place, considered as the cotangent bundle to momentum space. Momentum space is assumed to be a (not necessarily Riemannian) manifold Σ which possess a distinguished point $\underline{0}$ (the *origin*), a *metric* g and a *connection* Γ , which doesn't necessarily need to be metric.

Physical observables are related to intrinsic geometric concepts. The mass of a particle is measured by the geodesic distance of the particle's representing point in

momentum space from the origin,

$$d^2(p, \underline{0}) = m^2 . \quad (1)$$

This equation gives the dispersion relation. In this sense the metric of momentum space is related to the kinematical properties of a single particle.²

Dynamics, or the interaction between particles, is related to the connection, since the connection defines the composition law of momenta, $\oplus : \Sigma \times \Sigma \rightarrow \Sigma$ ³, through

$$\left. \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus_k q)_\rho \right|_{p=q=k} = -\Gamma_\rho^{\mu\nu}(k) , \quad (2)$$

where \oplus_k is the composition law “translated” at the point k

$$p \oplus_k q = \oplus_k ((\ominus k \oplus p) \oplus (\ominus k \oplus q)) . \quad (3)$$

The antisymmetric part of the connection is the Torsion, which measures the noncommutativity of the composition law

$$\left. \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus_k q - q \oplus_k p)_\rho \right|_{p=q=k} = -T_\rho^{\mu\nu}(k) , \quad (4)$$

while the curvature measures its nonassociativity

$$\left. \frac{\partial}{\partial p_{[\mu}} \frac{\partial}{\partial q_{\nu]}} \frac{\partial}{\partial r_\rho} ((p \oplus_k q) \oplus_k r - p \oplus_k (q \oplus_k r))_\sigma \right|_{p=q=r=k} = R_\sigma^{\mu\nu\rho}(k) . \quad (5)$$

The nonmetricity, defined from the metric and the connection as

$$N^{\mu\nu\rho} = \nabla^\rho g^{\mu\nu}(k) , \quad (6)$$

has been shown [1, 16] to be responsible for the leading order time-delay effect in the arrival of photons from distant sources, which is an effect that is currently under experimental verification [17].

The dynamics of interacting particles is obtained from a variational principle. In the case of a single vertex (interaction among n particles with momenta $p^j, j = 1 \dots n$) we need to minimize the following action:

$$S = \sum_j \left[\pm \int_{\sigma_0}^{\pm\infty} d\sigma \left(\left\{ -x_j^\mu \dot{p}_\mu^j + N_j (d^2(p^j, \underline{0}) - m^2) \right\} + z^\mu \mathcal{K}_\mu(p^1(\sigma_0), \dots, p^n(\sigma_0)) \right) \right] . \quad (7)$$

The \pm sign is chosen according to whether the j -th particle is outgoing or incoming. N_j and z^μ are Lagrange multipliers, but z^μ gives also the coordinates of the interaction point. $\mathcal{K}_\mu(p^1(\sigma_0), \dots, p^n(\sigma_0))$ may be any combination of all of the momenta in the vertex and gives the momentum conservation law, performed with the rules \oplus and \ominus . x_j^μ are the spacetime coordinates of the j -th particle, and the dot represents the derivative with respect to σ , an unphysical variable that parametrizes the trajectory of

² Note that the dispersion relation depends on the particular choice of coordinate system over the momentum space.

³ This law is assumed to admit a left and right inverse $\ominus : \Sigma \rightarrow \Sigma$, such that $\ominus p \oplus p = p \oplus (\ominus p) = \underline{0}$.

the system in phase space. σ_0 is an arbitrary value of σ at which the interaction is assumed to take place.

The constraints given by the variation with respect to the Lagrange multipliers N_n and z^μ are

$$d^2(p^j, \underline{0}) = m^2 , \quad (8)$$

which is the dispersion relation, and

$$\mathcal{K}_\mu(p^1(\sigma_0), \dots, p^n(\sigma_0)) = 0 , \quad (9)$$

which gives the conservation of energy and momentum in the interaction vertex.

The (bulk) equations of motion resulting from the minimization of the action are

$$\dot{p}_\mu^j = 0 , \quad \dot{x}^\mu_j = -N_j \frac{\partial}{\partial p_\mu^j} d^2(p^j, \underline{0}) . \quad (10)$$

The first equation expresses the conservation of particle momenta during free propagation. The second one implies that the spacetime worldlines are straight lines, and their speed is $v_k = \frac{\partial}{\partial p_k^j} d^2(p^j, \underline{0}) / \frac{\partial}{\partial p_0^j} d^2(p^j, \underline{0})$. In the case of special relativity the angular coefficient is the relativistic speed $v_k = p_k / \sqrt{p^2 + m^2}$. Note that the Lagrange multiplier N_j simply amounts to a normalization constant for the tangent vector to the trajectory, and has no physical meaning.

The boundary terms give the initial conditions

$$x_j^\mu(\sigma_0) = z^\mu \frac{\partial}{\partial p_\mu^j} \mathcal{K}_\mu(p^1, \dots, p^n) . \quad (11)$$

In the case of special relativity $\mathcal{K}_\mu(p^1, \dots, p^n) = \sum_j p_\mu^j$ and all the worldlines simply end up at the interaction point z^μ . If the nonlinearity of momentum space induces corrections to the composition law of momenta, then the worldlines will have slightly different endpoints. So the interaction doesn't appear local. Locality is recovered when the observer lays near z^μ , that is, the interaction takes place near the origin of the coordinate system, so that $z^\mu \simeq 0$ and $x_j^\mu(\sigma_0) \simeq 0 \ \forall j$. This expresses the principle of the relativity of locality.

To describe the physical picture perceived by different inertial observers, which are connected by (spacetime) translations and Lorentz transformations, we need the Poisson brackets of dynamical quantities with generators. Then the transformation law of coordinates is

$$x_j^{\mu'} = x_j^\mu + a^\nu \{ \mathcal{K}_\nu(p^1, \dots, p^n), x_j^\mu \} , \quad (12)$$

and it is easy to prove that, at the level of the equations of motion, this action effectively corresponds to translating *classically* the interaction point $z^{\mu'} = z^\mu + a^\mu$.

The translation generator in the case of more than one vertex is not known, and neither is the Lorentz transformation generator, even with a single vertex, if the momentum space is not simply a maximally symmetric space, where isometries are homomorphisms of the composition law:

$$\Lambda(p \oplus q) = \Lambda(p) \oplus \Lambda(q) . \quad (13)$$

From paper [1] it is not clear whether Lorentz transformations are a symmetries of the theory only in this simple case or also in more general cases.

We are going to use the results of this Section on the equations of motion to find out how the Lorentz transformations look like in κ -Poincaré, checking that they are indeed symmetries of the dynamics.

3. κ -Poincaré representation on momentum space

Hopf algebras possess in principle a sufficiently powerful structure to specify univocally a Cartan manifold, understood as a manifold with a metric and a connection, which does not have to be necessarily the Levi-Civita one, because torsion and nonmetricity are allowed.

We are indeed able to show that it is possible to represent the κ -Poincaré Hopf algebra as the algebra of coordinates and isometry operators over a curved momentum space. Then any algebraic structure characterizing this Hopf algebra has a corresponding geometric structure in the momentum space.

The bicrossproduct structure of κ -Poincaré, identified by Majid and Ruegg in [15], allows one to distinguish between the translation sector, whose generators we call P_μ , from the Lorentz sector, generated by boosts N_j and rotations R_k .

In this Section we are going to show the correspondence between the translation sector of κ -Poincaré and the coordinates over a manifold that can be interpreted as a momentum space. In Subsection 4.1 we will show then that the Lorentz sector of κ -Poincaré is an algebra of isometries on this momentum space (while in Section 5 we will show that the Lorentz sector of κ -Poincaré describes also the symmetries of particles dynamics). The translation sector can be interpreted as the algebra of functions over a manifold, which we identify with the momentum space Σ , such that the generators P_μ assign coordinates to points on the manifold in a certain coordinate system according to

$$P_\mu(p) = p_\mu , \quad (14)$$

where p represents a point on the manifold, and p_μ its coordinates.

A change of basis in the algebra generated by the P_μ corresponds to a change of coordinate system on the manifold:

$$P'_\mu = P'_\mu(P) \rightarrow P'_\mu(p') = p'_\mu , \quad \text{where } p'_\mu = p'_\mu(p_\nu) . \quad (15)$$

The coproduct map is related to the composition rule \oplus of momentum space points,

$$\Delta P_\mu(p, q) = (p \oplus q)_\mu . \quad (16)$$

Then from the coassociativity axiom of Hopf algebras:

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta , \quad (17)$$

the associativity of the momentum composition rule follows,

$$((p \oplus q) \oplus k) = (p \oplus (q \oplus k)) , \quad (18)$$

which, in turn, implies the flatness of the connection on the momentum manifold, cf. Eq. (5). The counit can be used to identify the coordinates of the origin of momentum space $\underline{0}$

$$P_\mu(\underline{0}) = \varepsilon(P_\mu) , \quad (19)$$

in a way that is compatible with the antipode axiom (μ is the multiplication of the Hopf algebra)

$$\mu \circ (S \otimes \text{id}) \circ \Delta = \mu \circ (\text{id} \otimes S) \circ \Delta = \mathbb{1} \varepsilon , \quad (20)$$

if we relate the antipode S with the inversion \ominus in the following way:

$$S(P_\mu)(p) = (\ominus p)_\mu . \quad (21)$$

In fact in this way we have

$$((S \otimes \text{id}) \circ \Delta)P_\mu(p, p) = ((\text{id} \otimes S) \circ \Delta)P_\mu(p, p) = ((\ominus p) \oplus p)_\mu = \varepsilon(P_\mu) . \quad (22)$$

Hopf algebra \mathcal{H}	Momentum space Σ
$\Delta : \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$	$\oplus : \Sigma \times \Sigma \rightarrow \Sigma$
$S : \mathcal{H} \rightarrow \mathcal{H}$	$\ominus : \Sigma \rightarrow \Sigma$
$\varepsilon : \mathcal{H} \rightarrow \mathbb{R}$	$\underline{0} : \mathbb{R} \rightarrow \Sigma$
generators P_μ	coordinate system p_μ
change of basis	diffeomorphism

Table 1. Duality between Hopf algebra and momentum space structures

We end up having a neat picture relating the geometric structures on momentum space introduced in Ref. [1], and the algebraic structure of the κ -Poincaré Hopf algebra, which we summarize in Table 1. The reason why we were able to get this for this is simple: the momentum composition rule \oplus , together with the origin $\underline{0}$ and the bilateral inverse $\ominus p$ equips the momentum space with an algebra loop structure: a group without the associativity axiom. If \oplus is associative, then we have a group. And in particular we have a Lie group, because its elements are points on a manifold. Now, it is well known [18] that abelian Hopf algebras are dual structures to Lie groups, and they are introduced as algebras of functions over the group. The duality allows to reconstruct everything about the group from the Hopf algebra and vice versa ⁴.

⁴With Hopf algebras, due to the coassociativity axiom, we are able only to describe momentum spaces with flat connections. If we wanted to find the algebraic structure associated to a momentum space with a non-associative composition law, we would have had to rely on Hopf algebroids, which are the dual structures to groupoids [19].

4. Geometric properties of the κ -Poincaré Hopf algebra

In the previous Section we have established a relation between the structures of the κ -Poincaré Hopf algebra and the ones of the associated momentum space. So now we can deduce the physical properties of particles living on this momentum space according to the framework of Relative Locality outlined in Section 2. To do this we need to describe in more details the geometric properties of the momentum space associated to the κ -Poincaré algebra, specifying the metric, which allows to deduce the dispersion relation of particles (see Eq. (1)), and the connection, with its nonmetricity and torsion, which are instead related to particle interactions (see eqs. (2), (4) and (6)).

For simplicity we will restrict the calculations to the 1+1 dimensional version of the κ -Poincaré algebra. The generalization to 3+1 dimensions is straightforward. To fix notation we report the main properties of κ -Poincaré in the bicrossproduct basis. The generators satisfy the commutation rules

$$[P, E] = 0, \quad [N, P] = \frac{\kappa}{2}(1 - e^{-2E/\kappa}) - \frac{1}{2\kappa}P^2, \quad [N, E] = P, \quad (23)$$

where E and P are the translation generators, and N is the boost generator. The coalgebra is

$$\begin{aligned} \Delta E &= E \otimes \mathbb{1} + \mathbb{1} \otimes E, & \Delta P &= P \otimes \mathbb{1} + e^{-E/\kappa} \otimes P, \\ \Delta N &= N \otimes \mathbb{1} + e^{-E/\kappa} \otimes N, \end{aligned} \quad (24)$$

and, finally, the antipodes and counits are

$$S(E) = -E, \quad S(P) = -e^{E/\kappa}P, \quad S(N) = -e^{E/\kappa}N, \quad (25)$$

$$\varepsilon(E) = \varepsilon(P) = \varepsilon(N) = 0. \quad (26)$$

4.1. Metric

It was stated several times in the literature that the κ -Poincaré describes a curved momentum manifold [15, 20], and this manifold has been claimed to be a de Sitter space of radius κ (see, in particular, [21, 22, 23]). Here we conclusively demonstrate that the metric is indeed that of a de Sitter space⁵, but in the next Sections we also show that the momentum space is not simply de Sitter, because it is endowed with torsion and nonmetricity, that change the connection in such a way that the curvature tensor is zero, unlike what happens in a de Sitter space with the Levi-Civita connection.

To find the metric we observe that the de Sitter line element in comoving coordinates

$$ds^2 = dE^2 - e^{2E/\kappa} dp^2 \quad (27)$$

is left invariant by the action of the κ -Poincaré boosts (23). In fact we can exponentialize the action of the boost generators on the momenta, in order to obtain the finite Lorentz

⁵During the final stages of preparation of this work we became aware, through a talk given by L. Smolin at the meeting Loops11, of an ongoing related project [24] which is reaching similar conclusions about the metric and connection for the κ -Poincaré Hopf algebra.

transformations, as done in [25] (also see [26]):

$$\begin{aligned} E' &= E + \kappa \log \left[\left(\cosh \xi/2 + \frac{p}{\kappa} \sinh \xi/2 \right)^2 - e^{-2E/\kappa} \sinh^2 \xi/2 \right] , \\ p' &= \kappa \frac{(\cosh \xi/2 + \frac{p}{\kappa} \sinh \xi/2) (\sinh \xi/2 + \frac{p}{\kappa} \cosh \xi/2) - e^{-2E/\kappa} \cosh \xi/2 \sinh \xi/2}{(\cosh \xi/2 + \frac{p}{\kappa} \sinh \xi/2)^2 - e^{-2E/\kappa} \sinh^2 \xi/2} . \end{aligned} \quad (28)$$

Then, plugging these expressions into the line element (27) one verifies that it is invariant:

$$ds'^2 = dE'^2 - e^{2E'/\kappa} dp'^2 . \quad (29)$$

We can also show that the metric is de Sitter in a constructive way, which will also provide a useful coordinate system to do the computations in the following Subsection. Consider the change of basis (remember that a change of basis in the algebra corresponds to a change of coordinates on the momentum manifold):

$$\begin{aligned} \eta_0 &= \kappa \sinh(E/\kappa) + e^{E/\kappa} P^2/2\kappa , \\ \eta_1 &= e^{E/\kappa} P . \end{aligned} \quad (30)$$

In these basis the algebra reduces to the Poincaré algebra

$$[\eta_0, \eta_1] = 0 , \quad [N, \eta_0] = \eta_1 , \quad [N, \eta_1] = \eta_0 , \quad (31)$$

but the transformation is not 1 to 1, because it can be inverted in two ways :

$$E_{\pm} = \kappa \log \left(\frac{\eta_0 \pm \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2}}{\kappa} \right) , \quad P_{\pm} = \frac{\kappa \eta_1}{\eta_0 \pm \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2}} . \quad (32)$$

This implies that the coalgebra in the new basis doesn't close,

$$\begin{aligned} \Delta \eta_1 &= \eta_1 \otimes e^{E/\kappa} + \mathbb{1} \otimes \eta_1 \\ \Delta \eta_0 &= \eta_0 \otimes e^{E/\kappa} + e^{-E/\kappa} \otimes \eta_0 + \frac{1}{\kappa} e^{-E/\kappa} \eta_1 \otimes \eta_1 , \end{aligned} \quad (33)$$

because we are not able to express the $e^{E/\kappa}$ factor in a unique way as a function of η_0 and η_1 .

However, if we introduce now the additional coordinate:

$$\eta_4 = \kappa \cosh(E/\kappa) - e^{E/\kappa} P^2/2\kappa , \quad (34)$$

we see that, since η_4 can be both positive and negative, it makes the change of basis invertible in a unique way:

$$E = \kappa \log \left(\frac{\eta_0 + \eta_4}{\kappa} \right) , \quad P = \frac{\kappa \eta_1}{\eta_0 + \eta_4} , \quad (35)$$

and the coalgebra then closes:

$$\begin{aligned} \Delta \eta_0 &= \frac{1}{\kappa} \eta_0 \otimes (\eta_0 + \eta_4) + \frac{\kappa}{\eta_0 + \eta_4} \otimes \eta_0 + \frac{\eta_1}{\eta_0 + \eta_4} \otimes \eta_1 , \\ \Delta \eta_1 &= \frac{1}{\kappa} \eta_1 \otimes (\eta_0 + \eta_4) + \mathbb{1} \otimes \eta_1 , \\ \Delta \eta_4 &= \frac{1}{\kappa} \eta_4 \otimes (\eta_0 + \eta_4) - \frac{\kappa}{\eta_0 + \eta_4} \otimes \eta_0 - \frac{\eta_1}{\eta_0 + \eta_4} \otimes \eta_1 . \end{aligned} \quad (36)$$

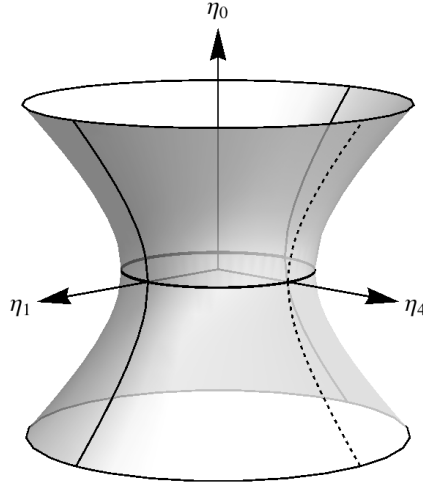


Figure 1. de Sitter space, with the η_a embedding coordinates.

We are then able to recognize the η_a ($a = 0, 1, 4$) generators⁶ as the ones associated (from a momentum space perspective) to the embedding coordinates of a two-dimensional de Sitter space of radius κ , since they satisfy the constraint:

$$\eta_0^2 - \eta_1^2 - \eta_4^2 = -\kappa^2 . \quad (37)$$

It is also possible to show that then N is the generator of the Lorentz subalgebra $so(1, 1) \in so(2, 1)$ of the isometries of the space:

$$[N, \eta_0] = \eta_1 , \quad [N, \eta_1] = \eta_0 , \quad [N, \eta_4] = 0 . \quad (38)$$

If now we induce the metric on the manifold defined by Eq. (37) from the flat metric in the ambient space, and we change back to the $\{E, p\}$ coordinates according to (32), we recover the metric (27).

4.2. Geodesics and particles dispersion relation

Now that we have the metric of the momentum space associated to the κ -Poincaré algebra, we can derive the physical properties of particles living on this momentum space, studying its geodesic equation, the connection, the torsion and the nonmetricity.

In the Relative Locality framework, the mass of a particle is given by the geodesic distance of the particle's representing point on momentum space from the origin. So each particle with mass m will live on a curve of constant geodesic distance from the origin, and the equation (1) relating mass and geodesic distance gives the particle's dispersion relation.

⁶The coordinates $\{\eta_0, \eta_1, \eta_4\}$ were first introduced in [21, 22, 23], where a relation between κ -Poincaré and de Sitter space was first conjectured.

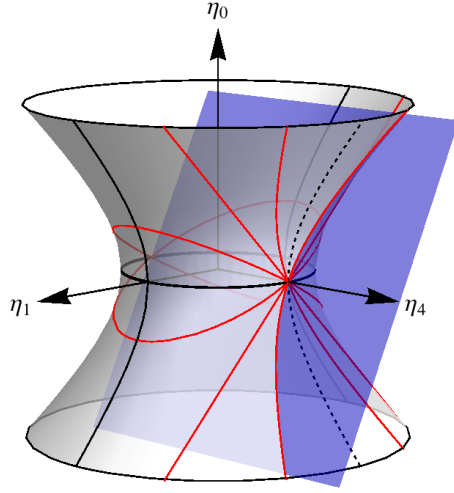


Figure 2. Geodesics (in red) from the origin of the η_a coordinate system in de Sitter space. They are given by the intersections with planes through the η_4 axis (in blue).

The geodesics in a de Sitter space are easily obtained in the embedding coordinates. They are given [27] by the intersection of the hyperboloid (37) with the planes passing through the center (in embedding coordinates: $\{\eta_0, \eta_1, \eta_4\} \equiv \{0, 0, 0\}$).

To write the dispersion relation for particles living on this de Sitter momentum space we need the geodesics that cross the origin $\underline{0}$, which in the η_a coordinates is the point $\{0, 0, \kappa\}$ ⁷. So all the geodesics we are interested in are given by the intersections with the planes that contain the η_4 axis (see Fig. 2).

The curves with constant geodesic distance from the origin are obtained through the intersection with the planes that are orthogonal to the η_4 axis (See Fig. 3). Their equation in embedding coordinates is:

$$\eta_4 = \kappa \cosh(d/\kappa) , \quad (39)$$

where d is the (constant) geodesic distance of the curve. Then, in the bicrossproduct coordinates E, P , the equation satisfied by constant geodesic distance curves is:

$$d(E, P) = \kappa \operatorname{arccosh} \left(\cosh(E/\kappa) - e^{E/\kappa} P^2 / 2\kappa^2 \right) . \quad (40)$$

So, according to the Relative Locality construction, this should be taken as the dispersion relation of particles whose momentum space has coordinates and isometries described by the κ -Poincaré algebra.

Notice that the usual proposal for the dispersion relation of κ -Poincaré is based on the Casimir [28, 29, 30] :

$$\square_\kappa = 4\kappa^2 \sinh^2(E/2\kappa) - e^{E/\kappa} P^2 \equiv 2\kappa(\eta_4 - \kappa) , \quad (41)$$

⁷Note how the essential role of the counit is here manifest: we know that the origin in the η_a coordinate system has these coordinates because $\varepsilon(\{\eta_0, \eta_1, \eta_4\}) = \{0, 0, \kappa\}$.

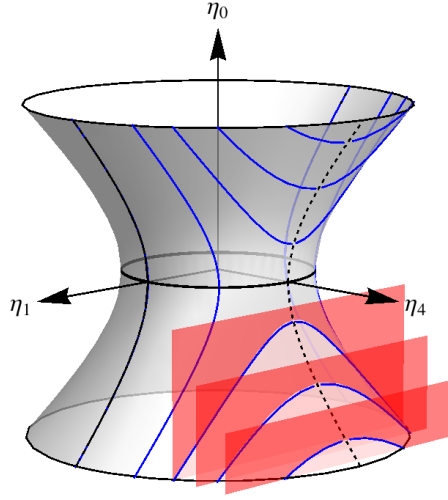


Figure 3. Curves of constant geodesic distance (in blue) from the origin of the η_a coordinate system. They are given by the intersections with the planes orthogonal to η_4 (in red).

which is a nonlinear function of our geodesic distance. The difference can be reabsorbed into a nonlinear redefinition of the mass.

An interesting observation following from this analysis is that the geodesic distance naturally selects a definition of mass as the rest energy of a particle. In fact, according to Eq. (1), the mass satisfies the relation:

$$\cosh(m/\kappa) = \cosh(E/\kappa) - e^{E/\kappa} P^2 / 2\kappa^2, \quad (42)$$

so that when $P = 0$ the dispersion relation gives $E = m$, and when $\kappa \rightarrow \infty$ the relation reduces to $E^2 - P^2 = m^2$. If instead, as it was customary to do in literature until now [28, 29, 30], one uses the Casimir (41) as the definition of the dispersion relation, then the rest energy and the mass would be related in a nonlinear way, $4\kappa^2 \sinh^2\left(\frac{E}{2\kappa}\right) = m^2$.

4.3. Connection, torsion, nonmetricity and composition of particles momenta

In Section 3 we have derived the properties of momenta composition rules from the properties of κ -Poincaré translation generators. On the other hand, in Section 2 we have stated that in the Relative Locality framework momenta composition rules are related to the geometric properties (connection, torsion) of the momentum space. Here we show explicitly this relation for the κ -Poincaré momentum space.

From the co-associativity of the coproduct of the κ -Poincaré generators, which means that the composition rule of momenta is associative (see Eq. (18)), it follows that the curvature vanishes⁸.

⁸The associativity of the composition law \oplus trivially implies that of the “translated” law \oplus_k , so the curvature vanish everywhere, according to Eq. (5).

The coproduct of the P and E generators, Eq. (24), can be used to write explicitly the “translated” composition law (3)

$$\begin{aligned} (p \oplus_k q)_0 &= p_0 + q_0 - k_0 , \\ (p \oplus_k q)_1 &= p_1 + e^{(k_0 - p_0)/\kappa} (q_1 - k_1) , \end{aligned} \quad (43)$$

which is needed to calculate the connection at an arbitrary point as in Eq. (2). Then the expressions of the connection and the torsion are

$$\Gamma_{\rho}^{\mu\nu} = - \frac{\partial}{\partial p_{\mu}} \frac{\partial}{\partial q_{\nu}} (p \oplus_k q)_{\rho} \Big|_{p=q=k} = \frac{1}{\kappa} \delta^{\mu}_0 \delta^{\nu}_1 \delta^1_{\rho} , \quad (44)$$

$$T_{\rho}^{\mu\nu}(k) = \frac{1}{\kappa} \delta^{[\mu}_0 \delta^{\nu]}_1 \delta^1_{\rho} . \quad (45)$$

From the connection and the metric we can derive the nonmetricity:

$$\begin{aligned} \nabla^{\rho} g^{\mu\nu} &= \partial^{\rho} g^{\mu\nu} + \Gamma_{\sigma}^{\mu\rho} g^{\sigma\nu} + \Gamma_{\sigma}^{\nu\rho} g^{\mu\sigma} \\ &= -\frac{1}{\kappa} \left(2 \delta^{\mu}_1 \delta^{\nu}_1 \delta^{\rho}_0 + \delta^{\mu}_0 \delta^{\nu}_1 \delta^{\rho}_1 + \delta^{\mu}_1 \delta^{\nu}_0 \delta^{\rho}_1 \right) e^{2E/\kappa} . \end{aligned} \quad (46)$$

As already noticed in Section 3, the connection is flat, in the sense that the Riemann tensor vanishes, due to the associativity of the composition law.

5. Lorentz transformations

The translation sector of κ -Poincaré can be interpreted as the algebra of functions over a curved momentum space, while, as we have shown in subsection 4.1, the Lorentz sector generates a subalgebra of isometries on the momentum space.

We have also seen that we can state a correspondence between the de Sitter momentum space defined by κ -Poincaré and the physical properties of particles living on it, but it is still not clear if the isometries on the momentum space represented by the boost generator actually correspond to transformations leaving the dynamics invariant. In particular the boost transformations need to be covariant also with respect to the composition of momenta.

In this Section we will be actually able to find this covariant action of boosts on composed momenta, showing that it requires the introduction of a sort of “back-reaction” of the momenta of each interacting particle on the boost rapidity acting on the other particles.

Let’s define the boost transformations in momentum space (in the bicrossproduct basis) as in Eq. (28)

$$\Lambda(\xi, p) = \left(\begin{array}{c} p_0 + \kappa \log \left[\left(\cosh \xi/2 + \frac{p_1}{\kappa} \sinh \xi/2 \right)^2 - e^{-2p_0/\kappa} \sinh^2 \xi/2 \right] \\ \kappa \frac{(\ch \xi/2 + \frac{p_1}{\kappa} \sh \xi/2)(\sh \xi/2 + \frac{p_1}{\kappa} \ch \xi/2) - e^{-2p_0/\kappa} \ch \xi/2 \sh \xi/2}{(\ch \xi/2 + \frac{p_1}{\kappa} \sh \xi/2)^2 - e^{-2p_0/\kappa} \sinh^2 \xi/2} \end{array} \right) ,$$

where ξ is the rapidity. Of course, since these transformations preserve the metric (cf. subsection 4.1), they also leave invariant the geodesic distance

$$d(\Lambda(\xi, p), \Lambda(\xi, q)) = d(p, q) . \quad (47)$$

Moreover they close an abelian group⁹,

$$\Lambda(\xi_1, \Lambda(\xi_2, p)) = \Lambda(\xi_1 + \xi_2, p) , \quad (48)$$

and they reduce to ordinary Lorentz transformations in the limit $\kappa \rightarrow \infty$:

$$\Lambda(\xi, p) = \begin{pmatrix} p_0 \cosh \xi + p_1 \sinh \xi \\ p_1 \cosh \xi + p_0 \sinh \xi \end{pmatrix} . \quad (49)$$

We want to find how these transformations act on composed momenta: this allows to determine how the momenta of various particles that interact in a vertex would appear to a boosted observer. The trivial solution, valid in special relativity, that each momentum transforms independently from the others, of course doesn't work here, since

$$\Lambda(\xi, p \oplus q) \neq \Lambda(\xi, p) \oplus \Lambda(\xi, q) . \quad (50)$$

A solution to this problem comes if we exploit a relation reported in [20]: momenta on which finite Lorentz transformations act turn out to have a “back-reaction” on them, since they change the rapidity in a momentum-dependent way, that is compatible with the coproduct of momenta, and with the action of Lorentz transformations on momenta themselves. This “back-reaction” is defined as the right action $\triangleleft : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$, that in bicrossproduct coordinates reads

$$\xi \triangleleft p = 2 \operatorname{arcsinh} \left(\frac{e^{-p_0/\kappa} \sinh \frac{\xi}{2}}{\sqrt{(\cosh \frac{\xi}{2} + \frac{p_1}{\kappa} \sinh \frac{\xi}{2})^2 - e^{-2p_0/\kappa} \sinh^2 \frac{\xi}{2}}} \right) . \quad (51)$$

This equation allows us to write the Lorentz transformation of two composed momenta as

$$\Lambda(\xi, q \oplus k) = \Lambda(\xi, q) \oplus \Lambda(\xi \triangleleft q, k) . \quad (52)$$

Then, if we call q' and k' the boosted momenta,

$$(q \oplus k)' = q' \oplus k' \quad , \quad q' = \Lambda(\xi, q) \quad , \quad k' = \Lambda(\xi \triangleleft q, k) , \quad (53)$$

and this law ensures that both the transformed momenta, q' and k' , are still on the mass-shell, because k' is just boosted, even if with a q -dependent rapidity:

$$d(k', 0) = d(\Lambda(\xi \triangleleft q, k), 0) = d(k, 0) . \quad (54)$$

Interestingly, the transformation law of any number of momenta participating to a vertex is highly asymmetric with respect to the exchange of momenta, and it keeps track of the order in which the momenta enter the vertex.

Considering the Lorentz transformation of three composed momenta, and applying the Lorentz transformation in the two possible orders (thanks to the associativity of \oplus we can forget about the brackets in the three-momenta sum)

$$\begin{aligned} \Lambda(\xi, p \oplus q \oplus k) &= \Lambda(\xi, p) \oplus \Lambda(\xi \triangleleft p, q) \oplus \Lambda(\xi \triangleleft p \triangleleft q, k) \\ &= \Lambda(\xi, p) \oplus \Lambda(\xi \triangleleft p, q) \oplus \Lambda(\xi \triangleleft p \oplus q, k) , \end{aligned} \quad (55)$$

⁹This is the only point in which the 3+1 dimensional case shows some complications with respect to the 1+1-d, because the Lorentz group in the 3+1-d case is nonabelian. However, there are no novelties with respect to Special Relativity here because the Lorentz subgroup is classical, and κ won't intervene in the composition law for rapidities

we deduce that the associativity of \oplus implies that the composition law of two consecutive actions of the momenta on the rapidity is:

$$\xi \triangleleft p \triangleleft q = \xi \triangleleft p \oplus q , \quad (56)$$

expressing the covariance of the right-action of momenta on rapidities with respect to the momenta composition law.

A few remarks on the boosts $\Lambda(\xi, p)$ and the back-reaction $\xi \triangleleft p$ we have used. As Majid observed [20] the boost and the back-reaction are defined for every value of the rapidity only if the momentum lies within the upper light cone $d(p, 0) \geq 0$. Otherwise for every other p there exists a finite critical boost ξ_c that makes $p_0 \rightarrow -\infty$, and after which the transformation $\Lambda(\xi, p)$ is not defined. Moreover, for every ξ there exist a critical curve in momentum space, which lies outside of the upper light-cone, on which $\xi \triangleleft p_c \rightarrow \pm\infty$, and after which the back-reaction is not defined.

These issues pose no problem for the classical theory, because one is only concerned with on-shell particles, which lie in the upper light cone. However, if one wanted to perform a “Feynman quantization” of Relative Locality, which means summing internal lines (*i.e.* lines lying between two vertices) over the whole momentum space, including off-shell parts, he would face this problem. We postpone the study of this issue to future works.

6. Equivalence between inertial observers

In the previous Section we have seen that when applied to interacting particles, Lorentz transformations act differently on each particle. The rapidity with which they act on each single particle depends on the momenta of the other particles which participate to the vertex and on their order. Here we show that physics is left invariant by these kind of Lorentz transformations, showing that the equations of motion (10) and (11) for the particles in the vertex are invariant.

Let’s consider a vertex with n interacting particles whose momenta composition law is $\mathcal{K} = k^1 \oplus \dots \oplus k^n$, which is boosted with rapidity parameter ξ , and let’s call

$$\xi^j \equiv \xi \triangleleft k^1 \oplus \dots \oplus k^{j-1} , \quad (57)$$

the rapidity with which the j -th moment boosts, so that

$$(k^j)' = \Lambda(\xi^j, k^j) , \quad (58)$$

and the conservation law transforms as

$$\mathcal{K}' = \Lambda(\xi^1, k^1) \oplus \dots \oplus \Lambda(\xi^n, k^n) \equiv \Lambda(\xi, \mathcal{K}) . \quad (59)$$

which means that it’s invariant ($\mathcal{K} = \underline{0} \Rightarrow \Lambda(\xi, \mathcal{K}) = \underline{0}$). From this and from the invariance of the geodesic distance under boosts we see that both the constraints (8) and (9) are invariant.

The particle coordinates will transform according to the transformation rule of covectors under diffeomorphisms:

$$(x'_j)^\mu = x_j^\nu \frac{\partial k_\nu^j}{\partial (k^j)_\mu'} = x_j^\nu \frac{\partial \Lambda(-\xi^i, (k^j)')_\nu}{\partial (k^j)_\mu'} . \quad (60)$$

The bulk equations of motion (10) are invariant under these transformations

$$\dot{k}_\mu^j = 0 \Rightarrow (\dot{k}^j)_\mu' = 0 , \quad \mathcal{K}_\mu = 0 \Rightarrow \mathcal{K}'_\mu = 0 , \quad (61)$$

$$\dot{x}_j^\mu = N_j \frac{\partial D(k, 0)}{\partial k_\mu^j} \Rightarrow (\dot{x}_j)'^\mu = N_j \frac{\partial D(k', 0)}{\partial (k^j)_\mu'} .$$

The invariance of the first equation is trivial: if all the k^j 's are constants, then also the transformed ones are so. The invariance of the second equation comes from the invariance of the origin, $\Lambda(\xi, \underline{0}) = \underline{0}$; and the last equation can be verified by direct computation.

Also the boundary equations (11) are invariant

$$x_j^\mu(s_0) = z^\nu \frac{\partial \mathcal{K}_\nu}{\partial k_\mu^j} \Rightarrow x_j'^\mu(s_0) = z'^\nu \frac{\partial \mathcal{K}'_\nu}{\partial (k^j)_\mu'} , \quad (62)$$

if the z 's transform as:

$$z'^\mu = z^\nu \frac{\partial \mathcal{K}_\nu}{\partial \mathcal{K}'_\mu} .$$

Notice that $\frac{\partial \mathcal{K}_\nu}{\partial \mathcal{K}'_\mu} = \Lambda^\nu_\mu$, and Λ^ν_μ is the classical Lorentz transformation of rapidity $-\xi^{10}$.

So, interestingly, it turns out that the vertex coordinates z^μ transform classically both under translations and under Lorentz transformations. Mathematically, this is a consequence of the fact that both these transformations are identical to the classical ones near the origin of momentum space¹¹, and the fact that the z^μ s transform under diffeomorphisms $p_\mu = f_\mu(p)$ as (see Ref. [16])

$$z'^\mu = z^\nu \left[\left(\frac{\partial f}{\partial p} \right)^{-1} \right]^\mu_{\nu} \Big|_{p=\underline{0}} , \quad (63)$$

where $\left[\left(\frac{\partial f}{\partial p} \right)^{-1} \right]^\mu_{\nu} \Big|_{p=\underline{0}}$ is the inverse diffeomorphism *calculated at the origin*.

7. z_μ coordinates and κ -Minkowski spacetime

One can use the κ -Poincaré connection (44) to calculate the parallel transport along a geodesic of an infinitesimal vector dq , living in the tangent space to the momentum space at the point p , from the point p to the origin [1],

$$(p \oplus dq)_\mu = p_\mu + dq_\nu (\tau_L)^\nu_\mu(p) , \quad \tau_L(p) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-p_0/\kappa} \end{pmatrix} , \quad (64)$$

¹⁰ $\frac{\partial \mathcal{K}_\nu}{\partial \mathcal{K}'_\mu}$ is equal to $\frac{\partial p_\nu}{\partial \Lambda(\xi, p)_\mu} \Big|_{p=0}$, which can be easily shown to be equal to a classical Lorentz transform.

¹¹ A fact that we relate to the “dual equivalence principle” formulated in [1], which states that locally the geometry of momentum space is that of Minkowski.

where τ_L is the parallel transport matrix, which relates the components of dq at p to its parallelly transported components at the origin.

In Ref. [1] the authors obtain the coordinates z_j^μ from the coordinates of the j -th particle with momentum p^j (therefore living in the tangent space to the momentum space at the point p), by parallelly transporting them along a geodesic toward the origin of the momentum space:

$$z_j^\mu \equiv x_j^\nu (\tau_L)^\mu{}_\nu (p^j) . \quad (65)$$

so that $z^0 = x^0$ and $z^1 = x^1 e^{-p_0/\kappa}$. These coordinates are not canonical as the x_j^μ 's, that close canonical Poisson brackets with the p_μ^j :

$$\{x_j^\mu, p_\nu^k\} = \delta^\mu{}_\nu \delta^k_j . \quad (66)$$

They instead close a Lie algebra among them

$$\{z_j^1, z_k^0\} = \frac{1}{\kappa} z_j^1 \delta_{jk} . \quad (67)$$

This algebra is the same satisfied by the coordinates of κ -Minkowski space, that is expected to be the noncommutative spacetime whose symmetries are described by κ -Poincaré (see [15, 31, 30]).

Eq. (67) indicates that relative locality may describe the “ $\hbar \rightarrow 0$ relics” of the κ -Minkowski noncommutative spacetime, which should be recovered upon quantization, transforming the Poisson brackets into commutators and the coordinates z_j^μ into operators \hat{z}_j^μ :

$$\{z_j^1, z_k^0\} = \frac{1}{\kappa} z_j^1 \delta_{jk} \rightarrow [\hat{z}_j^1, \hat{z}_k^0] = i \lambda \hat{z}_j^1 \delta_{jk} \quad (68)$$

where $\lambda = \hbar/\kappa$ is a length scale. This provides a hint for the physical interpretation of the κ -Minkowski algebra, as an algebra of functions over a noncommutative spacetime.

8. Conclusions and outlook

The κ -Poincaré Hopf algebra has been subject to an intense study since its discovery, almost 20 years ago. It attracted such a large interest because it provides a treatable example of “quantum geometry”, described through the language of symmetries. But until now a coherent picture has not been found, where its physical implications can be unambiguously determined, and a connection with the experiments can be made.

In this paper we established Relative Locality as the natural paradigm in which one should interpret the implications of κ -Poincaré for physics. This paradigm comes with a simple and coherent physical model, which allows to unambiguously determine the new effects that κ -Poincaré implies, thus finally allowing for the long-sought connection with the experiments.

We determined that κ -Poincaré describes a momentum space which metrically is de Sitter, with radius of curvature κ . This momentum space has a non-metric connection with zero curvature, and non-zero torsion and metricity.

Within this geometric interpretation we can find a natural proposal for particles dispersion relation, such that the particle's mass is equivalent to its rest energy.

These results lead to a fully workable model of point particles interacting with point-like, but “relatively-local”, interactions, with a non-symmetric and associative deformed conservation law of momentum.

We determined also the form of Lorentz transformations for this model, which allow us to confront the observations of different inertial observers. Interestingly, under boost transformations, on-shell momenta remain on-shell, with the same masses, but when we apply them to several interacting particles, the rapidity with which the momentum of each particle is boosted depends on the momenta of the other particles taking part to the interaction.

To achieve our results, we showed the equivalence between the Hopf algebra structures of κ -Poincaré and the geometric construction that realizes the principle of Relative Locality. This construction is directly applicable also to other Hopf algebras, and some analyses on the same lines are under development.

κ -Poincaré was initially obtained as the Inönü-Wigner contraction of another Hopf algebra, known as q-de Sitter [32, 33, 34, 35]. This algebra depends on two constants, H (dimensionful) and q (dimensionless), since it has been introduced as the q -deformation of the de Sitter algebra with radius H^{-1} . The contraction to κ -Poincaré corresponds to the limit $H \rightarrow 0$, $q \rightarrow 1$, where $H/\log q \rightarrow \kappa$. One can trade q for $\kappa = H/\log q$, which plays the role of an ultraviolet constant, while H is infrared [36]. This suggests an interpretation of q-de Sitter as the symmetry algebra of a system which possesses both curvature in momentum space and in spacetime. An interesting question is whether this situation can be fitted into the Relative Locality scheme.

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